

Calculus III: Coursework 8
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The deadline is 4PM on Wednesday Dec 5th. Hand-in is to the green box on the ground floor of the Maths building.

1. The function $u(x)$ obeys the ODE [40]

$$x^2 u'' + 2xu' + \lambda u/x^2 = 0, \quad (1)$$

where λ is a constant, and satisfies the conditions $u(1) = 0$ and $u \rightarrow 0$ as $x \rightarrow \infty$. Show that the ODE is of the Sturm-Liouville form, and state the appropriate standard form functions p , q and w .

By changing the independent variable to $t = 1/x$, and transforming the equation and boundary conditions, find a new Sturm-Liouville system in terms of t . Find its eigenvalues and eigenfunctions. Hence write down the eigenvalues and eigenfunctions of the original Sturm-Liouville system (1) and the corresponding orthogonality relation.

Answer

The equation can be written

$$(x^2 u')' + \lambda u/x^2 = 0$$

which is of the Sturm-Liouville form with $p = x^2$, $q = 0$ and $w = 1/x^2$.

In transforming, substituting the x^2 by $1/t^2$ is easy but we also have to rewrite the derivatives. For any function $f(x)$

$$\frac{df}{dx} = \frac{df}{dt} \frac{dt}{dx} \text{ (by the Chain rule) } = \frac{df}{dt} \left(\frac{-1}{x^2} \right) \Rightarrow -x^2 \frac{df}{dx} = \dot{f} \quad (2)$$

where as usual the dot means d/dt . This applies to $f = \dot{u}$ or $f = u'$ as well as to $f = u$. The original equation can be written

$$x^2(x^2 u')' + \lambda u = 0 \quad \Rightarrow \quad \ddot{u} + \lambda u = 0.$$

Alternatively, one can obtain $u' = -t^2 \dot{u}$ and $u'' = -t^2(-t^2 \dot{u})' = t^4 \ddot{u} - 2t^3 \dot{u}$ and substitute these into the original equation.

The boundary conditions become $u = 0$ at $t = 0$ ($x \rightarrow \infty$) and $t = 1$.

(From here on the derivation of eigenfunctions etc runs very similarly to the example in lectures, though the boundary conditions are a bit different.)

If $\lambda = -k^2 < 0$ the solution is $u_k = C_k \cosh kt + D_k \sinh kt$ and this cannot be 0 at both 0 and 1 unless $C_k = D_k = 0$.

If $\lambda = 0$ there is the special solution $u_0 = A_0 + B_0 t$ but this only satisfies the boundary conditions if $A_0 = B_0 = 0$.

If $\lambda = k^2 > 0$ the solution is $u_k = A_k \cos kt + B_k \sin kt$ where A_k and B_k are constants, and this will satisfy the boundary conditions (with $B_k \neq 0$) if $A_k = 0$ and $k = n\pi$ for integer n . This is the only case where $u_k \neq 0$.

So the eigenvalues are $\lambda = n^2\pi^2$ for integer n and the eigenfunctions are $u_{n^2\pi^2} = \sin n\pi t = \sin(n\pi/x)$.

The orthogonality relation, from the original equation, is

$$0 = \int_1^\infty \frac{u_{m^2\pi^2} u_{n^2\pi^2}}{x^2} dx = \int_1^\infty \frac{\sin(m\pi/x) \sin(n\pi/x)}{x^2} dx \quad (m \neq n)$$

(If uncertain about the general derivation, this can be checked by transforming to an integral in t and using a result from coursework 1.)

Marks: 4 for transforming correctly to S-L form, 1 each for p,q,w.

Transforming the equation 8 (correct chain rule use, 4 of those). Transforming boundary conditions 1 each.

Consideration of the different λ : correct method 5. Then 3 each for the hyperbolic and linear cases, and 6 for the actual eigenvalues and eigenfunctions in t . 2 for rewriting in terms of x . 4 for orthogonality.

2. Derive the result (stated in lectures) that the square wave defined by [12]

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

in the domain $[-\pi, \pi]$ and periodic with period 2π , has the Fourier series

$$f(x) = \frac{1}{2} + 2 \sum_{n \text{ odd}} \frac{\sin nx}{n\pi}.$$

Answer

An alert student may notice that $f(x) - \frac{1}{2}$ is an odd function and do its half range series. For those who do not spot this the calculations should read

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f \, dx = \frac{1}{\pi} \int_0^{\pi} 1 \, dx = \frac{1}{\pi} \pi = 1, \\ a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f \cos mx \, dx = \frac{1}{\pi} \int_0^{\pi} \cos mx \, dx = \frac{1}{\pi} [(\sin mx)/m]_0^{\pi} = 0, \\ b_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f \sin mx \, dx = \frac{1}{\pi} \int_0^{\pi} \sin mx \, dx = \frac{1}{\pi} [-(\cos mx)/m]_0^{\pi} = (1 - (-1)^m)/\pi m. \end{aligned}$$

Thus $b_m = 0$ for even m and $2/m\pi$ for odd m .

Marks: 4 for correct method, 2 each for a_0 and a_m and 4 for b_m done as above. Similarly for the ‘alert student’ method.

3. Express the following as Fourier series. [10]

(a) $\cos(x + \frac{\pi}{4})$

(b) $\cos^4 x$

[It is not necessary to do any integrals.]

Answer (a) $\cos(x + \frac{\pi}{4}) = \cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x$

(b) $\cos^4 x = \frac{1}{4}(1 + \cos 2x)^2 = \frac{1}{4}(1 + 2 \cos 2x + \cos^2 2x) = \frac{1}{4}(1 + 2 \cos 2x + \frac{1}{2}[1 + \cos 4x])$
 $\cos^4 x = \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x$

The example $\sin^4 x$ was done in lectures and $\cos^4 x - \sin^4 x = \cos^2 x - \sin^2 x = \cos 2x$ which provides a check or a cunning way to do the question.

Marks: 5 each part, suitable part marks for correct methods with wrong arithmetic.

4. Find the Fourier sine series for $f(x) = x^2$ on $(0, \pi)$. [14]
 What does the series converge to at $x = \pi$?

Answer

$$\begin{aligned} b_m &= \frac{2}{\pi} \int_0^\pi x^2 \sin mx dx \\ &= \frac{2}{\pi} \left(\left[\frac{-x^2 \cos mx}{m} \right]_0^\pi + \int_0^\pi \frac{2x \cos mx}{m} dx \right) \\ &= \frac{2}{\pi} \left(-\frac{\pi^2 \cos m\pi}{m} + \left[\frac{2x \sin mx}{m^2} \right]_0^\pi - \int_0^\pi \frac{2 \sin mx}{m^2} dx \right) \\ &= \frac{2}{\pi} \left(-\frac{\pi^2 (-1)^m}{m} - \left[\frac{2 \cos mx}{m^3} \right]_0^\pi \right) \\ &= \frac{2}{\pi} \left(-\frac{\pi^2 (-1)^m}{m} + \frac{2(1 - (-1)^m)}{m^3} \right) \end{aligned}$$

so on $(0, \pi)$ the series is

$$x^2 = \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{2(1 - (-1)^n)}{n^3} - \frac{\pi^2 (-1)^n}{n} \right) \sin nx$$

At $x = \pi$ there is a jump discontinuity. As $x \rightarrow \pi$ from below, $f \rightarrow \pi^2$ and since the series is odd, as $x \rightarrow -\pi$ from above, its value $\rightarrow -\pi^2$ so the series takes the average, which is 0.

Marks 10 for calculation of b_m (3 for correct method, 7 for accuracy, suitable part marks for integrations with errors). 4 for understanding and applying the convergence theorem.

5. In the lectures it was shown that the Fourier series for the function [24]

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ x & \text{if } 0 < x < \pi. \end{cases}$$

is

$$f = \frac{\pi}{4} - \sum_{k=1}^{\infty} \frac{2}{\pi(2k+1)^2} \cos(2k+1)x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx .$$

Write down the values of $f(-x)$ on $[-\pi, \pi]$ and the Fourier series for $f(-x)$. Using this result and the series for f itself, find the Fourier series for x and for $|x|$.

[As a check, the answer for $|x|$ should be

$$|x| = \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4}{\pi(2k+1)^2} \cos(2k+1)x .]$$

Use the Fourier series for x to derive

$$1 - \frac{1}{3} + \frac{1}{5} + \dots = \frac{\pi}{4} ,$$

(which was obtained in lectures from the square wave series) and use the same Fourier series and Parseval's relation [which you are reminded reads $\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) dx = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$ in the usual notation] to derive an infinite series for π^2 .

Answer

$$f(-x) = \begin{cases} -x & \text{if } -\pi < x < 0 \\ 0 & \text{if } 0 < x < \pi. \end{cases}$$

and, by swapping x and $-x$, its Fourier series is

$$f = \frac{\pi}{4} - \sum_{k=1}^{\infty} \frac{2}{\pi(2k+1)^2} \cos(2k+1)x - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx .$$

The Fourier series for $|x|$ is the sum of the two series, i.e.

$$|x| = \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4}{\pi(2k+1)^2} \cos(2k+1)x .$$

and the Fourier series for x is the difference of the two series

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx .$$

Evaluating the x series at $x = \pi/2$ gives

$$\frac{\pi}{2} = 2 \sum_{k=0}^{\infty} \frac{(-1)^{2k+1}}{2k+1} \sin((2k+1)\pi/2) = 2 \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1}$$

which easily gives the stated series.

Putting it into Parseval gives

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

which simplifies to

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \Rightarrow \pi^2 = 6 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Marks 3 for $f(-x)$ and 3 for its series. 2 for deriving the series for $|x|$ correctly (because they have the hint), 3 for the series for x .

Evaluating the x series 6 (2 for choosing a good point to evaluate at, 4 for calculation). Correct use of Parseval's relation 3, correct simplification 4. (Take 1 off if not rearranged to give π^2 .)

Note The relation between the result here and in example 7.6 in the lectures arises because

$$\sum_{n \text{ even}} \frac{1}{n^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}, \text{ so } \sum_{n \text{ odd}} \frac{1}{n^2} = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2},$$

and $\frac{3}{4} \frac{1}{6} = \frac{1}{8}$.